ADAPTING THE TIME STEP TO RECOVER THE ASYMPTOTIC BEHAVIOR IN A BLOW-UP PROBLEM

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ABSTRACT. The equation $u_t = \Delta u + u^p$ with homegeneous Dirichlet boundary conditions has solutions with blow-up if $p > 1$. An adaptive time-step procedure is given to reproduce the asymptotic behvior of the solutions in the numerical approximations. We prove that the numerical method reproduces the blow-up cases, the blow-up rate and the blow-up time. We also localize the numerical blow-up set.

1. INTRODUCTION.

We study the behavior of an adaptive time step procedure for the following parabolic problem

$$
\begin{cases} u_t = \Delta u + u^p & \text{in } \Omega \times [0,T), \\ u(x,t) = 0 & \text{on } \partial\Omega \times [0,T), \\ u(x,0) = u_0(x) > 0 & \text{on } \Omega. \end{cases} \tag{1}
$$

Where p is superlinear ($p > 1$) in order to have solutions with blow-up. We assume u_0 is regular and $\Omega \subset \mathbb{R}^d$ is a bounded smooth domain in order to guarantee that $u \in C^{2,1}$. A remarkable fact in this problem is that the solution may develop singularities in finite time, no matter how smooth u_0 is. For many differential equations or systems the solutions can become unbounded in finite time (a phenomena that is known as blow-up). Typical examples where this happens are problems involving reaction terms in the equation like (1) where a reaction term of power type is present and so this blow up phenomenum occurs in the sense that there exists a finite time T such that $\lim_{t\to T} \|u(\cdot,t)\|_\infty = +\infty$ for initial data large enough (see [22] [23] and the references therein). The blow-up set, which is defined as the set composed of points $x \in \Omega$ such that $u(x,t) \to +\infty$ as $t \to T$, is localized in small portions of Ω, in [24] is proved that the $(d-1)$ dimensional Hausdorff measure of the blow-up set is finite. The blow-up rate at these points is given by $u(x,t) \sim (T-t)^{-\frac{1}{p-1}}$, moreover

$$
\lim_{t\to T} (T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = C_p, \qquad C_p = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}
$$

(see [14],[15]).

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We remark that these results hold if p is subcritical ($p < \frac{d+2}{d-2}$ if $d \geq 3$). For supercritical p the solutions may present different behaviors. For that reason we assume p is subcritical along the paper, however the asymptotic properties of the numerical schemes described above hold for every $p > 1$. This is a difference between the continuous solutions and their approximations.

Since the solution u develops a singularity in finite time, we investigate what can be said about the asymptotic behavior (close to the blow-up time) of numerical approximations for this kind of problems. In [16] the authors analyze a semidiscrete scheme (keeping t continuous) in an interval, $\Omega = (0, 1)$. Here we generalize that analysis for several space dimensions an introduce the adaptive discretization in time.

For previous work on numerical approximations of (1) we refer to [2], [3], [6], [7], [8], [19], the survey [5] and references therein.

As a first step to introduce the method we propose a method of lines, that is: we discretize the space variable, keeping t continuous. In this stage we consider a general method with adequate assumptions. More precisely, we assume that for every $h > 0$ small (h is the parameter of the method), there exists a set of nodes $\{x_1, \ldots, x_N\} \subset \overline{\Omega}$ ($N = N(h)$), such that the numerical approximation of u at the nodes x_k, is given by

$$U(t) = (u_1(t), \ldots, u_N(t)).$$

That is $u_k(t)$ stands for an approximation of $u(x_k, t)$. We assume that U is the solution of the following ODE

$$MU'(t) = -AU(t) + MU^p, \tag{2}$$

with initial data given by $u_k(0) = u_0(x_k)$. In (2) and hereafter, all operations between vectors are understood coordinate by coordinate.

The precise assumptions on the matrices involved in the method are:

- (P1) M is a diagonal matrix with positive entries m_k.
- (P2) A is a nonnegative symmetric matrix, with nonpositive coefficients off the diagonal (i.e. $a_{ij} \leq 0$ if $i \neq j$) and $a_{ii} > 0$.
- (P3) $\sum_{k=1}^{N} a_{ik} \geq 0$.

As an example, we can consider a linear finite element approximation of problem (1) on a regular acute triangulation of Ω (see [9]). In this case, let V_h be the subspace of $H_0^1(\Omega)$ consisting of piecewise linear functions on the triangulation. We impose that the finite element approximation $u_h : [0, T_h) \to V_h$ verifies for each $t \in [0, T_h)$

$$\int_\Omega ((u_h)_t v)^I = -\int_\Omega \nabla u_h \nabla v + \int_\Omega ((u_h)^p v)^I,$$

for every $v \in V_h$. Here $(\cdot)^I$ stands for the linear Lagrange interpolate at the nodes of the mesh. These conditions imply that the vector $U(t)$, the values of $u_h(\cdot, t)$ at the nodes x_k, must verify a system of the form (2). In this case M is the lumped mass matrix and A is the stiffness matrix. The assumptions on the matrices M and A hold as we are considering an acute regular mesh. We observe that in this case $u_h = U^I$.

As another example, if Ω is a cube, $\Omega = (0,1)^d$, we can use a semidiscrete finite differences method to approximate the solution $u(x,t)$ obtaining an ODE system of the form (2).

We also need some kind of convergence result for the scheme, we will state the precise hypotheses concerning convergence in the statement of each theorem. Finally, in the Appendix we prove an L^∞ convergence theorem under the consistency assumption an give some examples. Now we state the two convergence hypotheses that we may need.

(H1) For every $\tau > 0$ $\|u - u_h\|_{L^\infty(\Omega\times[0,T-\tau])} \to 0$ as $h \to 0$

(H2) $\|u - u_h\|_{H_0^1(\Omega)}(t) \to 0$ as $h \to 0$ for a.e. t

Writing these equations explicitly we obtain the following ODE system,

$$m_k u_k'(t) = -\sum_{i=1}^{N} a_{ki} u_i(t) + m_k u_k^p(t), \ 1 \le k \le N, \tag{3}$$

with initial data $u_k(0) = u_0(x_k)$.

In [16] the authors studied these kind of schemes in one space dimension, obtaining results similar to those stated below.

Once obtained the ODE system, the next step is to discretize the time variable t. In [1] the authors suggest an adaptive time step procedure to deal with the heat equation with a nonlinear flux boundary condition. They analyze explicit Euler and Runge-Kutta methods, however all these methods have to deal with restrictions in the time-step. In this work we first analyze an explicit Euler method and next we introduce an implicit scheme in order to avoid the time-step restrictions. We use $U^j = (u_1^j, \dots, u_N^j)$ for the values of the numerical approximation at time t_j, and $\tau_j = t_{j+1} - t_j$. When we consider the explicit scheme, U^j is the solution of

$$\begin{aligned} &MU^{j+1} = MU^j + \tau_j\left(-AU^j + M(U^j)^p\right) \\ &U(0) = u_0^I, \end{aligned} \tag{4}$$

or equivalently, if we denote $\partial u_i^{j+1} = \frac{1}{\tau_j}(u_i^{j+1} - u_i^j)$

$$\begin{aligned} &m_i \partial u_i^{j+1} = -\sum_{k=1}^{N} a_{ki} u_i^j + m_i (u_i^j)^p, \ 1 \le i \le N \\ &u_i^0 = u_0(x_i), \qquad 1 \le i \le N+1. \end{aligned} \tag{5}$$

While for the implicit scheme U^j is the solution of

$$\begin{aligned} &MU^{j+1} = MU^j - \tau_j\left(AU^{j+1} + M(U^j)^p\right) \\ &U(0) = u_0^I. \end{aligned} \tag{6}$$

Note that the scheme is not totally implicit since the nonlinear source u^p is evaluated at time t^j while the discrete laplacian (A) is evaluated a time t^{j+1}. This mixture makes the scheme free of time-step restrictions while the explicit evaluation of $(U^j)^p$ avoids the problem of solving a nonlinear system in each step.

Now we choose the time steps $\tau_j = t_{j+1} - t_j$ in such a way that the asymptotic behavior of the discrete problem is similar to the continuous one. We will fix λ small and take

$$\tau_j = \frac{\lambda}{(w^j)^p},$$

where

$$w^j = \sum_{k=1}^{N} m_k u_k^j$$

This choice for the time step is inspired by [1], where the authors use an adaptive procedure similar to this. They adapt the time step in a similar way but using the maximum (L^∞-norm) instead of w^j (L^1-norm). In their problem the maximum is fixed at the right boundary node (i.e. $\|U^j\|_\infty = u_{N+1}^j$). In this problem, the maximum (the node k such that $u_k^j = \|U^j\|_\infty$) can move from one node to another as t goes forward. So it is better to deal with w^j since, for example, we can compute its derivative. Anyway, as we use a fixed mesh for the space discretization we can compare both norms. The motivation of this choice for the time-step is that, as will be shown, the behavior of w^j is given by

$$\partial w^{j+1} \sim (w^j)^p.$$

Hence, with our selection of τ_j we can obtain

$$w^{j+1} \sim w^j + \tau_j (w^j)^p = w^j + \lambda \sim w^0 + (j+1)\lambda,$$

and we obtain the asymptotic behavior of w^j, which is, as we will see, similar to the one for the continuous solution.

When we deal with the explicit scheme we will also require

$$\lambda < \min_{1 \le i \le N} \frac{m_i}{a_{ii}} (w^0)^p. \tag{7}$$

Then we study the asymptotic properties of the numerical schemes. We will say that a solution of (4) (or (6)) blows up if

$$\lim_{j\to\infty} \|U^j\|_\infty = \infty, \qquad \text{and} \qquad T_{h,\lambda} := \sum_{j=1}^{\infty} \tau_j < \infty,$$

we call $T_{h,\lambda}$ the blow-up time. To describe when the blow-up phenomena occurs in the discrete problem we introduce the following functional $\Phi_h : \mathbb{R}^N \to \mathbb{R}$.

$$\Phi_h(U) \equiv \langle AU, U\rangle - \langle \frac{1}{p+1} MU^{p+1}, ME\rangle,$$

where $E = (1, 1, \ldots, 1)$. The functional Φ_h is a discrete version of

$$\Phi(u)(t) \equiv \int_\Omega \frac{|\nabla u(s,t)|^2}{2}\, ds - \int_\Omega \frac{(u(s,t))^{p+1}}{p+1}\, ds.$$

This functional characterize the solutions with blow-up in the continuous problem: in [11], [15] it is proved that u blows up at time T if and only if $\Phi(u)(t) \to -\infty$

as $t \to T$. We prove a similar result for the discrete functional Φ_h and this allows as to prove that if the continuous solution has finite time blow-up its numerical approximation also blows up when the parameters of the method are small enough.

Next we study the asymptotic behavior for the numerical approximations of the solutions with blow-up and we find that they behave very similar to the continuous ones. In fact we find that if $u_{h,\lambda}$ is a numerical solution with blow-up at time $T_{h,\lambda}$ its behavior is given by

$$\max_{1\le i\le N} u_i^j \sim (T_{h,\lambda} - t^j)^{-1/(p-1)}.$$

We use the notation $f(j) \sim g(j)$ to mean that there exist constants $c, C > 0$ independent of j (but they may depend on h) such that

$$cg(j) \le f(j) \le Cg(j)$$

Moreover, the numerical schemes recover the constant C_p in the sense that

$$\lim_{j\to\infty} \max_{1\le i\le N} u_i^j (T_{h,\lambda} - t^j)^{1/(p-1)} = C_p.$$

The functional Φ_h is also useful to prove convergence of the blow-up times. Unfortunately we can only prove the convergence of an iterated limit,

$$\lim_{h\to 0} \lim_{\lambda\to 0} T_{h,\lambda} = T.$$

By means of the numerical blow-up rate we observe a propagation property for blow-up points. We prove that the nodes adjacent to those that blow-up as the maximum may also blow-up (opposite to the continuous problem), but they did it with a slower rate and the number of adjacent blow-up nodes is determined only by p and is independent of h and λ.

In other words, if we call $B^*(U)$ the set of nodes k such that

$$\lim_{j\to\infty} u_k^j (T_{h,\lambda} - t^j)^{1/(p-1)} = C_p,$$

the number of blow-up points outside $B^*(U)$ depends (explicitly) only on the power p.

We split the paper in two parts, in the first part we deal with the explicit scheme and in the second one we develop the analysis for the implicit method.

2. The explicit scheme

The main tool in our proofs is a comparison argument, so first of all we prove a lemma which states that this comparison argument holds. Since we need restrictions in the time-step to prove this lemma they are essential for every result in this section. That is not the case of the implicit scheme.

Definition 2.1. *We say that (Z^j) is a supersolution (resp.: subsolution) for* (4) *if verifies the equation with an inequality $\ge$ ($\le$) instead of an equality.*

Lemma 2.1. *Assume the time step verifies*

$$\tau_j < \min_{1\le i\le N} \frac{m_i}{a_{ii}}.$$

Let $(\overline{U}^j), (\underline{U}^j)$ *a super and a subsolution respectively for* (4) *such that* $\underline{U}^0 < \overline{U}^0$, *then* $\underline{U}^j < \overline{U}^j$ *for every* j.

Proof: Let $Z^j = \overline{U}^j - \underline{U}^j$, by an approximation argument we can assume that we have strict inequalities in (4), then (Z^j) verifies

$$\begin{array}{lll} M\partial Z^{j+1} & > & -AZ^j + M((\overline{U}^j)^p - (\underline{U}^j)^p, \\ Z^0 & > & 0. \end{array} \tag{1}$$

If the statement of the Lemma is false, then there exists a first time t^{j+1} and a node x_i such that $z_i^{j+1} \le 0$. At that time we have

$$z_i^{j+1} > (1 - \tau_j \frac{a_{ii}}{m_i}) z_i^j + \tau_j \left(- \sum_{k\neq i} a_{ik} z_k^j + (\overline{u}_i^j)^p - (\underline{u}_i^j)^p \right) \ge 0,$$

a contradiction. □

2.1. **When does the solution blow up.** In this section we find conditions under which the solution of (5) blows up, we begin with some lemmas.

As the matrix A is a symmetric (property (P2)), there exists a basis of eigenvectors for the following eigenvalue problem

$$A\phi_i = \lambda_i M \phi_i.$$

We call $\eta = \eta(h)$ the greatest eigenvalue of this problem, that is

$$0 \le \lambda_i \le \eta(h).$$

Lemma 2.2. *For every* $y \in \mathbb{R}^N$ *there holds*

$$\langle Ay, y\rangle \le \eta(h) \langle My, y\rangle.$$

Proof: As the matrix M defines a scalar product in $\mathbb{R}^N$, we can assume that the eigenvectors ϕ_i are normalized such that

$$\langle M\phi_i, \phi_j\rangle = \delta_{ij}.$$

Let $y \in \mathbb{R}^N, y = \sum_{i=1}^N \alpha_i \phi_i$, then

$$\begin{array}{lll} \langle Ay, y\rangle & = & \left\langle \sum_{i=1}^N \alpha_i \lambda_i M\phi_i, \sum_{j=1}^N \alpha_j \phi_j \right\rangle \\ & = & \sum_{i=1}^N \alpha_i^2 \lambda_i \langle M\phi_i, \phi_i\rangle \\ & \le & \eta(h) \langle My, y\rangle. \end{array}$$

□

Lemma 2.3. *If* $(U^j)_{j\ge 0}$ *is large enough, then blows up in finite time.*

Proof: Recall the definition of

$$w^j = \sum_{i=1}^{N} m_i u_i^j,$$

and observe that there exists constants $c, C > 0$ that depend only on h such that

$$c \max_{1\le k\le N} u_i^j \le w^j \le C \max_{1\le i\le N} u_i^j.$$

Now we observe that we can choose $j \ge j_0$ in order to get w^j large enough to verify

$$\begin{aligned} w^{j+1} &= w^j - \tau_j \sum_{k=1}^{N}\sum_{i=1}^{N} a_{ik} u_k^j + \tau_j \sum_{i=1}^{N} m_i (u_i^j)^p \\ &\ge w^j - \frac{1}{2}\tau_j \sum_{i=1}^{N} m_i (u_i^j)^p + \tau_j \sum_{i=1}^{N} m_i (u_i^j)^p \\ &\ge w^j + \frac{\tau_j}{2} \sum_{i=1}^{N} m_i (u_i^j)^p \\ &\ge w^j + c\tau_j (w^j)^p \\ &= w^j + c\lambda. \end{aligned}$$

Applying this inequality inductively we obtain

$$w^j \ge w^{j_0} + c\lambda(j - j_0) \ge cj,$$

hence $w^j \to \infty$ as $j \to \infty$, it remains to check that $\sum \tau_j < \infty$, to do that we observe

$$\tau_j = \frac{\lambda}{(w^j)^p} \le \frac{\lambda}{(w^0 + jc\lambda)^p},$$

and

$$\sum_{j=1}^{\infty} \frac{\lambda}{(w^0 + j(c\lambda))^p} \le \int_0^{\infty} \frac{\lambda}{(w^0 + cs\lambda)^p}\, ds < \infty.$$

This completes the proof. □

Remark 2.1. *In the course of the proof of the above Lemma we showed not just that w^j blows up, we also proved $w^j \ge cj$.*

Now we are going to prove the reverse inequality to obtain the asymptotic behavior of $\|U^j\|_\infty$.

Lemma 2.4. *If (U^j) is unbounded then*

$$\|U^j\|_\infty \sim w^j \sim j$$

Proof: The relation $\|U^j\|_\infty \sim w^j$ is trivial since they define equivalent norms in $\mathbb{R}^N$. So we just have to prove $w^j \le Cj$. To do that we observe that

$$\begin{aligned} w^{j+1} &= w^j - \tau_j \sum_{k=1}^{N}\sum_{i=1}^{N} a_{ik} u_k^j + \tau_j \sum_{i=1}^{N} m_i (u_i^j)^p \\ &\le w^j + \tau_j \sum_{i=1}^{N} m_i (u_i^j)^p \\ &\le w^j + C\tau_j (w^j)^p \\ &= w^j + C\lambda. \end{aligned}$$

We apply this inequality inductively again to get

$$w^j \le w^0 + C\lambda j \le Cj,$$

as we wanted to prove. □

Theorem 2.1. *Assume the time step τ_j verifies the restriction*

$$\tau_j < \frac{2}{p(w^{j+1})^{p-1} + \eta(h)}. \tag{2}$$

Then positive solutions of (5) blow up if there exists j_0 such that $\Phi_h(U^{j_0}) < 0$.

We remark that the condition $\Phi_h(U^{j_0}) < 0$ is similar to the one for the blow-up phenomena in the continuous problem, in fact for the continuous problem this is a necessary condition.

Proof: First we observe that $\Phi_h(U^j)$ decreases with j, in order to do that we take inner product of (4) with $U^{j+1} - U^j$ to obtain

$$\begin{aligned} 0 &= \langle \frac{1}{\tau_j} M(U^{j+1} - U^j) + AU^j - M(U^j)^p, U^{j+1} - U^j \rangle \\ &= \tau_j \langle M\partial U^{j+1}, \partial U^{j+1}\rangle + \Phi_h(U^{j+1}) - \Phi_h(U^j) - \frac{1}{2}\langle AU^{j+1}, U^{j+1}\rangle \\ &+ \langle AU^j, U^{j+1}\rangle - \frac{1}{2}\langle AU^j, U^j\rangle - \frac{1}{2}\langle Mp(\xi^j)^{p-1}, (U^{j+1} - U^j)^2\rangle. \end{aligned}$$

Hence we obtain,

$$\begin{aligned} &\Phi_h(U^{j+1}) - \Phi_h(U^j) \\ &\le \tau_j \Big(\tau_j \frac{p(w^{j+1})^{p-1}}{2} - 1\Big)\langle M\partial U^{j+1}, \partial U^{j+1}\rangle + \frac{\tau_j^2}{2}\langle A\partial U^{j+1}, \partial U^{j+1}\rangle \\ &\le \tau_j \Big(\tau_j \frac{p(w^{j+1})^{p-1}}{2} + \frac{\eta(h)\tau_j}{2} - 1\Big)\langle M\partial U^{j+1}, \partial U^{j+1}\rangle \le 0, \end{aligned}$$

due the restriction in the time step τ_j and Lemma 2.2. Actually $\Phi_h(U^{j+1}) < \Phi_h(U^j)$ unless (U^j) is independent of j. So, Φ_h is a Lyapunov functional for (4). Next we observe that the steady states of (4) have positive energy. Let $(W^j) = W$ be a stationary solution of (4), then

$$0 = -AW + MW^p.$$

Multiplying by $W/2$ we obtain,

$$\begin{array}{rcl} 0 & = & -\frac{1}{2}\langle AW, W\rangle + \frac{p+1}{2}\frac{1}{p+1}\langle MW^p, W\rangle \\ & \geq & -\Phi_h(W). \end{array}$$

Now, assume (U^j) is a bounded solution of (4), then there exists a convergent subsequence of (U^j) that we still denote (U^j). Its limit W is a steady state with positive energy.

As $\Phi_{h,\lambda}(U^j)$ decreases and there exists j_0 such that $\Phi_{h,\lambda}(U^{j_0}) < 0$ then $\Phi_{h,\lambda}(W) < 0$, a contradiction. We conclude that (U^j) is unbounded and by Lemma 2.3 has finite time blow-up. □

Corollary 2.1. *Assume the time-step restriction of the above theorem and the convergence hypotheses* (H1), (H2)*. Let u_0 an initial data for* (1) *such that u blows up in finite time T. Then $u_{h,\lambda}$ blows up in finite time $T_{h,\lambda}$ for every h, $\lambda = \lambda(h)$ small enough. Moreover*

$$\lim_{h\to 0}\lim_{\lambda\to 0} T_{h,\lambda} = T$$

Remark 2.2. *If the fully-discrete method converges in $H_0^1(\Omega)$ a.e. t then λ can be chosen independent of h.*

Proof: We observe that if u blows up in finite time T then (see [11], [15])

$$\Phi(u)(t) \equiv \int_\Omega \frac{|\nabla u(s,t)|^2}{2}\, ds - \int_\Omega \frac{(u(s,t))^{p+1}}{p+1}\, ds \to -\infty \qquad (t \nearrow T).$$

This implies that there exists a time $t_0 < T$ with $\Phi(u)(t_0) < 0$. Now we use the convergence of $u_{h,\lambda}$ to u_h in $[0, t_0]$ and the convergence hypothesis (H1) to see that

$$\lim_{h\to 0}\lim_{\lambda\to 0} \Phi_{h,\lambda}(u_{h,\lambda})(t_0) = \Phi(u)(t_0).$$

So for $h, \lambda = \lambda(h)$ small enough we get $\Phi_{h,\lambda}(u_{h,\lambda})(t_0) < 0$ and so, by the above Theorem (U^j) blows up.

Now we turn our attention to the blow-up times. In [16] it is proved that the blow-up time of the semi-discrete solutions (solutions of (2)), that we are going to denote T_h, converges, as $h \to 0$, to T. In that work the poof is given for $\Omega = (0,1)$ and a finite element method. Despite for this case the proof is very similar, we sketch it for the sake of completeness. In [16] is proved that if the continuous solution blows up then for every h small enough the semidiscrete scheme also does. Hence we can assume that the semidiscrete solution $U(t)$ is large enough in order to verify

$$\begin{array}{c} \frac{d}{dt}\langle MU(t), U(t)\rangle = 2\langle MU'(t), U(t)\rangle = \\ 2\langle -AU(t), U(t)\rangle + 2\langle MU^p(t), U(t)\rangle = \\ -4\Phi_h(U(t)) + \frac{2(p-1)}{p+1}\langle MU^p(t), U(t)\rangle \geq \end{array}$$

$$4|\Phi_h(U(t))| + \frac{2(p-1)}{p+1}(\langle MU(t), U(t)\rangle)^{\frac{p+1}{2}}.$$

Integrating between t_0 and T_h we obtain

$$(T_h - t_0) \leq \frac{C}{(-\Phi_h(U(t_0)))^{\frac{p-1}{p+1}}}. \tag{3}$$

where C depends only on p.

Given $\varepsilon > 0$, we can choose M large enough to ensure that

$$\left(\frac{C}{M^{\frac{p-1}{p+1}}}\right) \leq \frac{\varepsilon}{2}.$$

As u blows up at time T we can choose $\tau < \frac{\varepsilon}{2}$ such that

$$-\Phi(u(\cdot, T-\tau) \geq 2M.$$

If h is small enough,

$$-\Phi_h(U(T-\tau)) \geq M,$$

and hence

$$T_h - (T-\tau) \leq \left(\frac{C}{(-\Phi_h(U(T-\tau)))^{\frac{p-1}{p+1}}}\right) \leq \left(\frac{C}{M^{\frac{p-1}{p+1}}}\right) \leq \frac{\varepsilon}{2}.$$

Therefore,

$$|T_h - T| \leq |T_h - (T-\tau)| + |\tau| < \varepsilon.$$

We have proved $\lim_{h\to 0} T_h = T$, so we just have to prove that for fixed h

$$\lim_{\lambda\to 0} T_{h,\lambda} = T_h.$$

To do that we observe that from Lemma 2.3 we know that there exists j_0, that does not depend on λ such that for $j \geq j_0$

$$w^j \geq w^{j_0} + c\lambda(j - j_0),$$

hence

$$\begin{aligned} T_{h,\lambda} - t^j &= \sum_{l=j+1}^{\infty} \tau_l = \sum_{l=j+1}^{\infty} \frac{\lambda}{(w^l)^p} \\ &\leq \sum_{k=j+1}^{\infty} \frac{\lambda}{(w^{j_0} + c\lambda(l-j_0))^p} \leq \int_j^{\infty} \frac{\lambda}{(w^{j_0} + c\lambda(s-j_0))^p}\, ds \\ &= \frac{1}{c}\int_{w^{j_0}+c\lambda(j-j_0)}^{\infty} \frac{1}{s^p}\, ds \leq \frac{1}{c}\int_{w^{j_0}}^{\infty} \frac{1}{s^p}\, ds. \end{aligned} \tag{4}$$

This holds for any j_0 large enough and for every $j \geq j_0$. In particular we get

$$T_{h,\lambda} - t^j \leq \frac{1}{c}\int_{w^j}^{\infty} \frac{1}{s^p}\, ds.$$

This inequality has a great meaning. It says that if w^j is large, then t^j is close to blow up (independent of λ). So now, given $\varepsilon > 0$ we can choose K large enough in order to have

$$\frac{1}{c}\int_K^\infty \frac{1}{s^p}\,ds \le \frac{\varepsilon}{3}, \qquad K^{-p} < \frac{\varepsilon}{3}.$$

Next we choose $\tau < \frac{\varepsilon}{3}$ such that $\sum m_k u_k(T_h - 2\tau) \ge 2K$ (remember that $(u_1(t), \dots, u_N(t))$ is the solution of the semidiscrete scheme). For $\lambda = \lambda(h,\tau)$ small enough we get, from (H2), that $w^j \ge K$ for every j such that $T_h - 2\tau \le t^j \le T_h - \tau$. We choose one of those j and compute

$$\begin{aligned} |T_{h,\lambda} - T_h| &\le |T_{h,\lambda} - t^j| + |t^j - T_h| \\ &\le \frac{1}{c}\int_K^\infty \frac{1}{s^p}\,ds + 2\tau \\ &\le \varepsilon \end{aligned}$$

□

2.2. **Blow-up rate.** In this section we study the asymptotic behavior of numerical solutions with blow-up.

Theorem 2.2. *Let $u_{h,\lambda}$ a solution with blow-up at time $T_{h,\lambda}$, then*

$$\max_{1\le i\le N} u_i^j \sim (T_{h,\lambda} - t^j)^{-1/(p-1)}.$$

Moreover

$$\lim_{j\to\infty} \max_{1\le i\le N} u_i^j (T_{h,\lambda} - t^j)^{1/(p-1)} = C_p = \left(\frac{1}{p-1}\right)^{1/(p-1)}.$$

We want to remark that this is the behavior of the continuous solutions with blow-up.

Proof: We know from Lemma 2.3 that $w^j = \sum m_i u_i^j$ verifies

$$w^{j+1} \ge w^j + c\tau_j (w^j)^p,$$

so we have

$$\begin{aligned} (T_{h,\lambda} - t^j) &= \sum_{k=j+1}^\infty \tau_j = \sum_{k=j+1}^\infty \frac{\lambda}{(w^j)^p} \\ &\le \int_j^\infty \frac{\lambda}{(w(s))^p}\,ds. \end{aligned}$$

(5)

Here $w(s)$ is the linear interpolant of $(w^j = w(j))$, hence for $j \le s \le j+1$ we have $w'(s) = w^{j+1} - w^j \ge c\lambda$, and so

$$\int_j^\infty \frac{\lambda}{(w(s))^p}\,ds \le \int_{w^j}^\infty \frac{\lambda}{cv^p\lambda}\,dv \le \frac{1}{c(p-1)}\left(\frac{1}{w^j}\right)^{p-1},$$

or equivalently

$$\max_{1\le i\le N} u_i^j \le Cw^j \le C(T_{h,\lambda}-t^j)^{-1/(p-1)}.$$

The inverse inequalities can be handled in a similar way to obtain

$$\max_{1\le i\le N} u_i^j \sim w^j \sim (T_{h,\lambda}-t^j)^{-1/(p-1)}.$$

Next we recover the constant C_p in the asymptotic behavior of the numerical solution, to do that we will change variables but first we prove a short lemma.

Lemma 2.5.

$$\lim_{j\to\infty} \frac{T_{h,\lambda}-t^j}{T_{h,\lambda}-t^{j+1}} = 1$$

Proof:

$$\begin{aligned} 1 \le \frac{T_{h,\lambda}-t^j}{T_{h,\lambda}-t^{j+1}} &= \frac{\sum_{k=j+1}^{\infty}\tau_k}{\sum_{k=j+2}^{\infty}\tau_k} = 1+\frac{\tau_{j+1}}{\sum_{k=j+2}^{\infty}\tau_k} \\ &\le 1+\frac{\lambda/(w^{j+1})^p}{C/(w^{j+1})^{p-1}} \to 1. \end{aligned}$$

□

Now we change variables, in a way inspired by [14],[16]. Let (Y^j) be defined by

$$y_i^j = u_i^j (T_{h,\lambda}-t^j)^{1/(p-1)} \qquad 1\le i\le N. \tag{6}$$

In the sequel of the proof we will use Δy_i^{j+1} to denote

$$\frac{y_i^{j+1}-y_i^j}{\tau_j/(T_{h,\lambda}-t^j)},$$

This can be thought as $\tau_j/(T_{h,\lambda}-t^j)$ to be the time step in the new variables. With this notation the new variables verify

$$\begin{aligned} m_i\Delta y_i^{j+1} &= -\frac{(T_{h,\lambda}-t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda}-t^j)^{\frac{1}{p-1}}}(T_{h,\lambda}-t^j)\sum_{i=1}^{N} a_{ki}y_i^j \\ &\quad +m_i\frac{(T_{h,\lambda}-t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda}-t^j)^{\frac{1}{p-1}}}(y_i^j)^p \\ &\quad +\frac{(T_{h,\lambda}-t^j)m_iu_i^j}{\tau_j}\left((T_{h,\lambda}-t^{j+1})^{\frac{1}{p-1}}-(T_{h,\lambda}-t^j)^{\frac{1}{p-1}}\right), \\ y_i^0 &= (T_{h,\lambda})^{1/(p-1)}u_0(x_i), \qquad 1\le i\le N+1. \end{aligned} \tag{7}$$

Now assume there exists $\varepsilon>0$ and a subsequence that we still denote (y_i^j) such that $y_i^j > C_p+\varepsilon$ for some $i=i(j)$. Then for those y_i^j we have

$$(y_i^j)^p - \frac{1}{p-1} y_i^j > \frac{3\delta}{m_i}.$$

We also know from the blow-up rate that the new variables y_i^j are bounded and so, applying Lemma 2.5 we obtain for j large enough

$$\begin{aligned} m_i \Delta y_i^{j+1} &\geq -\delta + m_i \left((y_i^j)^p - \frac{1}{p-1} y_i^j \right) \quad (8) \\ &\quad + m_i (y_i^j)^p \left(\frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}}} - 1 \right) \quad (9) \\ &\geq \delta. \quad (10) \end{aligned}$$

This means that actually $y_i^j > C_p + \varepsilon$ for every j large enough and consequently (8) is verified for all those j. So y_i^j is unbounded, a contradiction.

If we assume $y_i^j < C_p - \varepsilon$ arguing along the same lines we obtain that y_i^j verifies

$$\begin{aligned} m_i \Delta y_i^{j+1} &\leq \delta + m_i \frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}}} \left((y_i^j)^p \frac{1}{p-1} y_i^j \right) \\ &\quad + \frac{m_i}{p-1} y_i^j \left(\frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}}} - \frac{(T_{h,\lambda} - \xi^j)^{\frac{1}{p-1}-1}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}-1}} \right) \\ &\leq 2\delta + C \left((y_i^j)^p - \frac{1}{p-1} y_i^j \right). \quad (11) \end{aligned}$$

This shows that either $y_i^j \to 0$ as $j \to \infty$ or $m_i \Delta y_i^{j+1} < -\delta$, which means that y_i^j is not bounded from below (this is not possible).

We conclude that if y_i^j does not go to zero, then it goes to C_p, as the blow-up rate implies that for every j

$$\max_{1 \leq i \leq N} y_i^j \geq c,$$

we have

$$\lim_{j \to \infty} \max_{1 \leq i \leq N} y_i^j = \lim_{j \to \infty} \max_{1 \leq i \leq N} (T_{h,\lambda} - t^j)^{1/(p-1)} u_i^j = C_p,$$

as we wanted to prove. □

2.3. **Blow-up set.** Now we turn our attention to the blow-up set. In order to do that we consider the set $B^*(U)$ composed of those nodes that blow-up as the maximum (a precise definition of $B^*(U)$ is given in the introduction) and we study the behavior of the adjacent nodes, then we repeat the procedure with these nodes.

Definition 2.2. *We define the graph with vertices in the nodes and say that two different nodes are connected if and only if $a_{ij} \neq 0$. We consider the usual distance between nodes measured as a graph, see [18]. Finally, we denote by $d(k)$ the distance of the node x_k to $B^*(U)$ also measured as a graph.*

We prove that u_k blows up if and only if $d(k) \leq K$ where K depends only on p,

Theorem 2.3. *Let $B^*(U)$ be the set of nodes, $\{x_k\}$, such that*

$$u_k^j \sim (T_{h,\lambda} - t^j)^{-\frac{1}{p-1}}.$$

Then the blow-up propagates in the following way, let $p > 1$ and $K \in \mathbb{N}_0$ such that $\frac{K+2}{K+1} < p \leq \frac{K+1}{K}$ (K is the integer part of $1/(p-1)$). Then the solution of (4) blows up exactly at K nodes near $B^*(U)$. More precisely,

$$u_k^j \to +\infty \qquad \Longleftrightarrow \qquad d(k) \leq K.$$

Moreover, if $d(k) \leq K$, the asymptotic behaviour of $(u_k^j)_{j\geq 1}$ is given by

$$u_k^j \sim (T_{h,\lambda} - t^j)^{-\frac{1}{p-1}+d(k)},$$

if $p \neq \frac{K+1}{K}$ and if $p = \frac{K+1}{K}$, $d(k) = K$

$$u_k^j \sim \ln(T_{h,\lambda} - t^j).$$

We want to remark that more than one node can go to infinity, but the asymptotic behavior imposes $\frac{u_k^j}{u_i^j} \to 0$ $(j \to \infty)$ if $d(k) > d(i)$.

Proof of Theorem 1.4 We want to show that the blow-up propagates K nodes around $B^*(U)$, we begin with a node x_k such that $d(k) = 1$. We claim that the behavior of u_k^j is given by

$$u_k^j \sim \begin{cases} j^{-p+2} & \text{if} \quad p < 2 \\ \ln j & \text{if} \quad p = 2 \\ C & \text{if} \quad p > 2, \end{cases}$$

to prove that we will show that

$$w_A^j = A \sum_{s=1}^{j} s\tau_{s-1},$$

which has the behavior described above, can be used as super and subsolution for an equation verified by u_k^j choosing A appropriately.

We observe that u_k^j satisfies

$$\begin{aligned} m_k \partial u_k^{j+1} &= -\sum_{i=1}^{N} a_{ik} u_i^j + m_k (u_k^j)^p \\ &\sim C_1 (\max_{1\leq i\leq N} u_i^j) - C_2 u_k^j + C_3 m_k (u_k^j)^p. \end{aligned}$$

This means that there exists constants $c_i, C_i > 0$, $i = 1, 2, 3$ such that for j large enough

$$\partial u_k^{j+1} \leq C_1 j - C_2 u_k^j + C_3 (u_k^j)^p \tag{12}$$

and

$$\partial u_k^{j+1} \geq c_1 j - c_2 u_k^j + c_3 (u_k^j)^p \tag{13}$$

Now we observe that if A and j are large enough, w_A^j verifies

$$\begin{array}{rcl} \partial w_A^j & = & A(j+1) \\ & \geq & C_1 j - C_2 w_A^j + C_3 (w_A^j)^p, \end{array}$$

since $(w_A^j)^p / j \to 0$ as j goes to infintiy. Hence w_A^j is a supersolution for (12) and so

$$u_k^j \leq w_A^j.$$

On the other hand if we choose A small we get

$$\begin{array}{rcl} \partial w_A^j & = & A(j+1) \\ & \leq & c_1 j - c_2 w_A^j + c_3 (w_A^j)^p, \end{array}$$

Hence now we can use w_A^j as a subsolution for (15) to handle the other inequality. Therefore

$$u_k^j \sim w_A^j.$$

We observe that if $p < 2$ the node x_k is a blow-up node and we also have the blow-up rate for this node ($u_k^j \sim j^{-p+2}$). If $p > 2$ this node is bounded. Next we assume $p < 2$ (if $p > 2$ it is easy to prove that every node k with $d(k) \geq 1$ is bounded) and we are going to find the behavior of a node, that we still denote k, such that $d(k) = 2$.That is, it is not adyacent to $B^*(U)$ and it is adyacent to a node wich has the behavior j^{-p+2}.

Now let

$$w_A^j = A \sum_{s=1}^{j} \tau_s s^{-p+2},$$

and observe that u_k^j verifies

$$m_k \partial u_k^{j+1} = -\sum_{i=1}^{N} a_{ik} u_i^j + m_k (u_k^j)^p \sim C_1 (j^{-p+2}) - C_2 u_k^j + C_3 m_k (u_k^j)^p.$$

This means that there exists constants $c_i, C_i > 0$, $i = 1, 2, 3$ such that for j large enough

$$\partial u_k^{j+1} \leq C_1 j^{-p+2} - C_2 u_k^j + C_3 (u_k^j)^p \tag{14}$$

and

$$\partial u_k^{j+1} \geq c_1 j^{-p+2} - c_2 u_k^j + c_3 (u_k^j)^p \tag{15}$$

Now we observe that if A and j are large enough, w_A^j verifies

$$\begin{array}{rcl} \partial w_A^j & = & A(j+1)^{-p+2} \\ & \geq & C_1 j^{-p+2} - C_2 w_A^j + C_3 (w_A^j)^p, \end{array}$$

since $(w_A^j)^p/j^{-p+2} \to 0$ as j goes to infintiy. Hence w_A^j is a supersolution for (14) and so

$$u_k^j \leq w_A^j.$$

On the other hand if we choose A small we get

$$\begin{array}{rcl} \partial w_A^j & = & A(j+1)^{-p+2} \\ & \leq & c_1 j^{-p+2} - c_2 w_A^j + c_3 (w_A^j)^p, \end{array}$$

Now we can use w_A^j as a subsolution for (15) to handle the other inequality. So

$$u_k^j \sim w_A^j.$$

We observe that if $p < 3/2$ the node x_k is a blow-up node and we also have the blow-up rate for this node ($u_k^j \sim j^{-2p+3}$). If $p > 3/2$ this node is bounded. In the case $p < 3/2$ we repeat this procedure inductively to obtain the theorem. □

3. The implicit scheme

In order to avoid the time step restrictions we now introduce an implicit scheme and prove that similar properties can be observed. However we have to remark that near the blow-up time the adaptive procedure forces the time step to be much smaller than the space discretization parameter h. This suggest that an adequate method could be to begin at time zero with the implicit scheme in order to avoid time-step restrictions and, as when the solution increases the time step is reduced, if the solution has finite time blow-up, then there will be a moment such that the time step restriction for the explicit scheme will be verified. From that time one can continue either with the explicit or with the implicit scheme. As in the explicit scheme section we begin with the comparison Lemma.

Lemma 3.1. *Let* $(\overline{U}^j), (\underline{U}^j)$ *a super and a subsolution respectively for* (6) *such that* $\underline{U}^0 < \overline{U}^0$, *then* $\underline{U}^j < \overline{U}^j$ *for every* j.

Proof: Let $Z^j = \overline{U}^j - \underline{U}^j$, we assume that we have strict inequalities in (6), then (Z^j) verifies

$$\begin{array}{lcl} M\partial Z^{j+1} & > & -AZ^{j+1} + M((\overline{U}^j)^p - (\underline{U}^j)^p), \\ Z^0 & > & 0. \end{array} \tag{1}$$

If the statement of the Lemma is false, then there exists a first time t^{j+1} and a node x_i such that $z_i^{j+1} \leq 0$. There we have

$$z_i^{j+1} > z_i^j - \tau_j \frac{a_{ii}}{m_i} z_i^{j+1} - \sum_{k \neq i} \frac{a_{ik}}{m_i} z_k^{j+1} + (\overline{u}_i^j)^p - (\underline{u}_i^j)^p \geq 0,$$

a contradiction.

3.1. **When does the numerical solution blow up.**

Lemma 3.2. *If* $(U^j)_{j\geq 0}$ *is large enough, then blows up in finite time. Moreover*

$$\|U^j\|_\infty \sim w^j \sim j$$

Proof:

$$\begin{aligned} w^{j+1} &= w^j - \tau_j \sum_{k=1}^N \sum_{i=1}^N a_{ik} u_k^{j+1} + \tau_j \sum_{i=1}^N m_i (u_i^j)^p \\ &\leq w^j + \tau_j \sum_{i=1}^N m_i (u_i^j)^p \\ &\leq w^j + C\tau_j (w^j)^p \\ &= w^j + C\lambda. \end{aligned}$$

Hence $w^j \leq Cj$. To prove the inverse inequality we observe that

$$\begin{aligned} w^{j+1} &= w^j - \tau_j \sum_{k=1}^N \sum_{i=1}^N a_{ik} u_k^{j+1} + \tau_j \sum_{i=1}^N m_i (u_i^j)^p \\ &\geq w^j - \tau_j C_1 w^{j+1} + \tau_j C_2 (w^j)^p \end{aligned}$$

(2)

that is

$$(1 + C_1\tau_j) w^{j+1} \geq w^j + C_2 \tau_j (w^j)^p. \tag{3}$$

Now we look for a subsolution of (3) of the form $z^j = \Gamma j$. This sequence verifies

$$(1 + C_1\tau_j) z^{j+1} = z^j + \Gamma C_1 \tau_j j + \Gamma(1 + C_1\tau_j) \leq z^j + C_2\tau_j (z^j)^p$$

if Γ is small enough. As the discrete maximum principle holds for this equation we obtain

$$w^j \geq z^j = \Gamma j.$$

This completes the proof. □

Theorem 3.1. *Positive solutions of* (5) *blow up if there exists* j_0 *such that* $\Phi_h(U^{j_0}) < 0$.

Proof: Again we first observe that $\Phi_h(U^j)$ decreases with j, to do that we take inner product of (6) with $U^{j+1} - U^j$ to obtain

$$\begin{aligned}
0 &= \langle \frac{1}{\tau_j} M(U^{j+1}-U^j) + AU^{j+1} - M(U^j)^p, U^{j+1}-U^j\rangle \\
&= \tau_j \langle M\partial U^{j+1}, \partial U^{j+1}\rangle + \Phi_h(U^{j+1}) - \Phi_h(U^j) + \frac{1}{2}\langle AU^{j+1}, U^{j+1}\rangle \\
&\quad -\langle AU^j, U^{j+1}\rangle + \frac{1}{2}\langle AU^j, U^j\rangle + \frac{p}{2}\langle M(\xi^j)^{p-1}, (U^{j+1}-U^j)^2\rangle.
\end{aligned}$$

Hence we obtain,

$$\begin{aligned}
\Phi_h(U^{j+1}) - \Phi_h(U^j) &= -\tau_j \langle M\partial U^{j+1}, \partial U^{j+1}\rangle - \frac{\tau_j^2}{2}\langle A\partial U^{j+1}, \partial U^{j+1}\rangle \\
&\quad -\frac{p}{2}\langle M(\xi^j)^{p-1}, (U^{j+1}-U^j)^2\rangle \\
&\leq 0.
\end{aligned}$$

This implies that Φ_h is a Lyapunov functional for (6). We observe that the steady states of (6) are the same of (4), so they have positive energy. Now, assume (U^j) is a bounded solution of (6), then there exists a convergent subsequence of (U^j) that we still denote (U^j). Its limit W is a steady state with positive energy.

As $\Phi_h(U^j)$ decreases and there exists j_0 such that $\Phi_h(U^{j_0}) < 0$ then $\Phi_{h,\lambda}(W) < 0$, a contradiction. We conclude that (U^j) is unbounded and by Lemma 3.2 has finite time blow-up. □

Corollary 3.1. *Assume the convergence hypotheses* (H1), (H2). *Let u_0 an initial data for* (1) *such that u blows up in finite time T. Then $u_{h,\lambda}$ also blows up in finite time $T_{h,\lambda}$ for every h, $\lambda = \lambda(h)$ small enough. Moreover*

$$\lim_{h\to 0}\lim_{\lambda\to 0} T_{h,\lambda} = T.$$

Proof: If u blows up in finite time T then (see [11],[15])

$$\Phi(u)(t) \equiv \int_\Omega \frac{|\nabla u(s,t)|^2}{2}\, ds - \int_\Omega \frac{(u(s,t))^{p+1}}{p+1}\, ds \to -\infty \qquad (t \nearrow T).$$

Hence there exists a time $t_0 < T$ with $\Phi(u)(t_0) < 0$. Now we use the convergence hypothesis (H1) and the convergence of $u_{h,\lambda}$ to u_h in $[0, t_0]$ to see that

$$\lim_{h\to 0}\lim_{\lambda\to 0} \Phi_{h,\lambda}(u_{h,\lambda})(t_0) = \Phi(u)(t_0).$$

So for $h, \lambda(h)$ small enough we get $\Phi_{h,\lambda}(u_{h,\lambda})(t_0) < 0$ and so (U^j) blows up. The convergence of the blow-up times is obtained like in the explicit scheme.

□

Next we turn our attention to the blow-up rate of the discrete solutions.

3.2. **Blow-up rate.**

Theorem 3.2. *Let $u_{h,\lambda}$ a solution with blow-up at time $T_{h,\lambda}$, then*

$$\max_{1\leq i\leq N} u_i^j \sim (T_{h,\lambda} - t^j)^{-1/(p-1)}.$$

Moreover

$$\lim_{j\to\infty} \max_{1\le i\le N} u_i^j (T_{h,\lambda} - t^j)^{1/(p-1)} = C_p = \left(\frac{1}{p-1}\right)^{1/(p-1)}.$$

Proof: The first part of the proof is the same as the one for the explicit scheme so we assume we have proved

$$\|U^j\|_\infty \sim (T_{h,\lambda} - t^j)^{-\frac{1}{p-1}},$$

and we are going to recover the constant C_p. We change variables as in the explicit scheme, let (Y^j) be defined by

$$y_i^j = u_i^j (T_{h,\lambda} - t^j)^{1/(p-1)} \qquad 1 \le i \le N. \tag{4}$$

In the sequel of the proof we will use Δy_i^{j+1} to denote

$$\frac{y_i^{j+1} - y_i^j}{\tau_j/(T_{h,\lambda} - t^j)},$$

this can be thought as $\tau_j/(T_{h,\lambda})$ to be the time step in the new variables. With this notation the new variables verify

$$\begin{aligned} m_i \Delta y_i^{j+1} &= -(T_{h,\lambda} - t^j) \sum_{i=1}^{N} a_{ki} y_i^{j+1} + m_i \frac{(T_{h,\lambda} - t^{j+1})^{1/(p-1)}}{(T_{h,\lambda} - t^j)^{1/(p-1)}} (y_i^j)^p \\ &- m_i u_i^j ((T_{h,\lambda} - t^j)^{1/(p-1)} - (T_{h,\lambda} - t^{j+1})^{1/(p-1)}), \end{aligned} \tag{5}$$

$$y_i^0 = T_{h,\lambda}^{1/(p-1)} u_0(x_i). \tag{6}$$

If we assume that there exists $\varepsilon > 0$ and a subsequence such that $y_i^j > C_p + \varepsilon$ for some $i = i(j)$. Then for those y_i^j, as they are bounded, if j is large enough we have

$$\begin{aligned} \Delta y_i^{j+1} &\ge -\delta + m_i \frac{(T_{h,\lambda} - t^{j+1})^{1/(p-1)}}{(T_{h,\lambda} - t^j)^{1/(p-1)}} \left((y_i^j)^p - \frac{1}{p-1} y_i^j \right) \\ &\quad + \frac{1}{p-1} y_i^j \left[1 - \frac{(T_{h,\lambda} - t^{j+1})^{\frac{1}{p-1}-1}}{(T_{h,\lambda} - t^j)^{\frac{1}{p-1}-1}} \right] \\ &\ge \delta. \end{aligned} \tag{7}$$

This means that actually $y_i^j > C_p + \varepsilon$ for every j large enough and consequently (7) is verified for all those j. So y_i^j is unbounded, a contradiction.

The case where there exists an infinite number of y_i^j such that $y_i^j < C_p - \varepsilon$ is very similar. So we can conclude that as $j \to \infty$ either $y_i^j \to 0$ or $y_i^j \to C_p$. Now we use the blow-up rate to obtain

$$\lim_{j\to\infty} \max_{1\le i\le N} y_i^j = \lim_{j\to\infty} \max_{1\le i\le N} (T_{h,\lambda} - t^j)^{1/(p-1)} u_i^j = C_p,$$

as we wanted to prove. $\square$

3.3. **Blow-up set.** The propagation property for the blow-up nodes holds for the implicit scheme and its proof is very similar. We do not include it.

4. Appendix

In this appendix we prove that if the general method considered for the space discretization is consistent (see below) then the totally discrete method converges in the L^∞ norm. We perform the proofs for the explicit scheme, they can be extended to the implicit one.

Definition 4.1. *Let w be a regular solution of*

$$\begin{array}{ll} w_t = \Delta w + f(x,t) & \text{in } \Omega \times (0,T), \\ w = 0 & \text{on } \partial\Omega \times (0,T). \end{array}$$

We say that the scheme (2) *is consistent if for any $t \in (0, T - \tau)$ it holds*

$$m_i w_t(x_i,t) = -\sum_{k=1}^{N} a_{ik} w(x_k,t) + m_i f(x_i,t) + \rho_{i,h}(t), \tag{1}$$

and there exists a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\max_i \frac{|\rho_{i,h}(t)|}{m_i} \le \rho(h), \qquad \text{for every } t \in (0, T-\tau),$$

with $\rho(h) \to 0$ if $h \to 0$. The function ρ is called the modulus of consistency of the method.

If we consider for example a finite differences scheme in a cube $\Omega = (0,1)^d \subset \mathbb{R}^d$. Then the modulus of consistency can be taken as $\rho(h) = Ch^2$.

Theorem 4.1. *Let u be a regular solution of* (1) *($u \in C^{2,1}(\overline{\Omega} \times [0, T-\tau])$) and $u_{h,\lambda}$ the numerical approximation given by* (5) *then there exists a constant C depending on $\|u\|$ in $C^{2,1}(\overline{\Omega} \times [0, T-\tau])$ such that*

$$\|u - u_{h,\lambda}\|_{L^\infty([0,T-\tau],L^2(\Omega))} \le C(\rho(h) + \lambda).$$

Proof:

We define the error functions

$$e_i^j = u(x_i, t_j) - u_i^j.$$

By (1), these functions verify

$$m_i \partial e_i^{j+1} = -\sum_{k=1}^{N} a_{ik} e_k^j + m_i (u^p(x_i,t_j) - (u_i^j)^p) + \rho_i(h) + C m_i \lambda,$$

where C is a bound for $\|u_{tt}\|_{L^\infty(\Omega\times[0,T-\tau])}$. Let

$$t_0 = \max\{t:\ t < T-\tau,\ \max_i \max_{t_j<t} |e_i^j| \le 1\}.$$

We will see by the end of the proof that $t_0 = T-\tau$ for h,λ small enough. In $[0,t_0]$ we have

$$m_i \partial e_i^{j+1} = -\sum_{k=1}^{N} a_{ik} e_k^j + m_i p(\xi_i^j)^{p-1} e_i^j + \rho_i(h) + C m_i \lambda,$$

hence, in $[0,t_0]$, $E^j = (e_1^j, ..., e_N^j)$ satisfies

$$M\partial E^{j+1} \le -AE^j + KME^j + (\rho(h) + C\lambda)M1^t. \qquad (2)$$

Let us now define $W^j = (w_1^j, \dots, w_N(t))$, which will be used as a supersolution.

$$w_i^j = e^{(2K+1)t_j}(\|e(0)\|_\infty + \rho(h) + C\lambda).$$

It is easy to check that W^j verifies

$$M\partial W^{j+1} > -AW^j + KMW^j + (\rho(h) + C\lambda)M1^t,$$

here K is the Lipchitz constant for $f(u) = u^p$ in $[0, \|u(\cdot, T-\tau)\|_{L^\infty}]$. Hence W^j is a supersolution for (2), and by Lemma 2.1 we get

$$e_i^j \le e^{(2K+1)t_j}(\|e^0\|_{L^\infty(\Omega)} + \rho(h) + C\lambda), \qquad t_j \in [0,t_0].$$

Arguing along the same lines with $-E^j$, we obtain

$$|e_i^j| \le e^{(2K+1)T}(\|E(0)\|_\infty + \rho(h) + C\lambda) \le C(\rho(h) + \lambda), \qquad t_j \in [0,t_0],$$

by our hypotheses on the convergence of the initial data. Using this fact, since $\rho(h)$ goes to zero, we get that $|e_i^j| \le 1$ for every $t_j \in [0, T-\tau]$ for every h,λ small enough. Therefore $t_0 = T-\tau$ for h,λ small enough. This proves the convergence of the scheme. In fact we have that for every $h < h_0, \lambda < \lambda_0$

$$\max_j \max_{1\le i\le N} |u_i^j - u(x_i,t_j)| \le C(\rho(h) + \lambda).$$

□

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